Functional Integral Approach to C^*-algebraic Quantum Mechanics

J. LaChapelle

Abstract

The algebraic approach to quantum mechanics has been key to the development of the theory since its inception, and the approach has evolved into a mathematically rigorous C^*-algebraic formulation of the axioms. Conversely the functional approach in the form of Feynman path integrals is far from mathematically rigorous: Nevertheless, path integrals provide an equally valid and useful formulation of the axioms of quantum mechanics. The two approaches can be merged by employing a recently developed notion of functional integration that allows to construct functional integral representations of C^*-algebras. The merger is a hybrid formulation of the axioms of quantum mechanics in which topological groups play a leading role.

1 Introduction

The axioms of quantum mechanics (QM) are typically realized using either of two complementary strategies; the functional approach as embodied in the Feynman path integral or the C^*-algebraic approach.[1] The axioms themselves can be roughly classified as kinematical and dynamical. On the kinematical side sit the notions of Hilbert space of states, self-adjoint operators and the Born rule; while on the dynamical side sit the unitary evolution operator, the evolution equation, and observation/measurement.

The Feynman path integral[1] lies at the heart of the functional approach. The application of path integrals in standard QM is well-understood. However, being a formal object, it is often difficult to apply in more general settings with surety. Nevertheless, it is difficult to overstate the value of the physical intuition inherent in this approach.

The algebraic realization of the axioms was initiated by von Neumann[2] and extended in [3],[4]. It essentially culminated in the Gelfand-Naimark-Segal construction and the Gelfand-Naimark theorem allowing the kinematical QM axioms to be formulated in terms of certain linear functionals on C^*-algebras. By now the structure has been vastly elaborated and we refer to [5], [6] for details and more references.

[1] Our comments are restricted to QM where the functional and algebraic approaches are well-understood and enjoy equal status. Unfortunately, the algebraic approach to quantum field theory (QFT) for interacting fields is notoriously problematic. On the other hand, the functional integral approach, with its S-matrix interpretation, is quite developed and successful and consequently dominates in QFT. We do not address QFT in this paper.

The upshot is there exists a powerful and well-developed C^*-algebraic framework on which to base the kinematics of QM.

But the dynamics — both evolution and observation — requires further input and interpretation: Dynamics is naturally described by specifying a one-parameter group of $*$-automorphisms of the algebra, while the observation/measurement issue remains unresolved (or at least controversial) and open to interpretation. Dynamical evolution leads to the study of derivations on the algebra; and eventually, correspondence with classical physics fixes the nature of the derivations — though, famously, not uniquely in general. This somewhat awkward inclusion of dynamics in the algebraic formulation has stimulated further work to develop a single encompassing structure.

One approach to implement both the kinematics and dynamics of QM under one algebraic roof utilizes *crossed products* (which are reviewed from a physics perspective in [7]–[9] and rigorously developed in [10]). Crossed products originated in the work of Mackey[11]-[13] on representation theory. The utility of crossed products in the context of QM is: (i) they provide a single algebraic structure built from the original C^*-algebra encoding kinematics and the $*$-automorphism group encoding dynamics, and (ii) they may be used to realize $*$-representations of the integrated Heisenberg equation on an associated Hilbert space. Stated precisely, there is a one-to-one correspondence between a covariant representation of a dynamical system (which, in particular, encodes the integrated Heisenberg equation of motion) and non-degenerate representations of the crossed product ([10] § 2.2–2.4). Accordingly, crossed products provide a convenient implementation of the C^*-algebraic approach to QM — both kinematics and dynamics.

Of course there exists a bridge between the functional and algebraic approaches; but, other than supplying a translation dictionary, it is of limited use. This is unfortunate because it effectively separates the formal/heuristic appeal of path integrals and the rigorous mathematical development of C^*-algebras.

The purpose of this paper is to describe a substantial upgrade to that bridge. Our main tool, the functional Mellin transform[14], provides a generalization of the standard Gaussian path integral and contains crossed products as a subclass. *Under suitable conditions, the space of Mellin integrable functionals is a C^*-algebra, and the functional Mellin transform is a $*$-representation.* In particular, functional Mellin transforms allow to represent quantum operators, their traces, and their determinants as functional integrals.

The motivation for introducing functional integrals into the algebraic approach should be obvious: The formal and/or heuristic application of path integrals in the functional approach is quite useful, and one hopes to enjoy similar benefits by applying functional integral techniques in the C^*-algebra setting. The resulting formulation is a generalization of the Feynman path integral representation of a quantum system that is determined solely by an underlying topological group and its irreducible representations.

2 Quantization

The two major ingredients required for algebraic quantization are a C^*-algebra and a group of $*$-automorphisms of the algebra. Incorporating functional integrals into the picture pays immediate dividends: it strongly suggests that a single object— a topological group — generates the entire structure. In consequence, given a (generally non-abelian) topological group, the functional integral framework to which we adhere (see appendix A) provides a C^*-algebra of equivariant functions along with its associated inner automorphisms. The simple idea is that this structure models both the kinematics and dynamics of a closed quantum system.

2.1 Preliminaries

Let's set the stage for quantization. The kinematic input is: (i) some C^*-algebra $\mathfrak{A}_{\mathrm{L}}$ *equipped with a Lie bracket structure* whose self-adjoint elements encode observable properties of some quantum system, (ii) an associated Hilbert space $\mathcal{H}$ that furnishes a physically relevant representation $\pi : \mathfrak{A}_{\mathrm{L}} \to L_B(\mathcal{H})$ where $L_B(\mathcal{H})$ is the algebra of bounded linear operators on $\mathcal{H}$, and (iii) suitable linear functionals to account for the Born rule.

Now, a particularly interesting subset of elements of the algebra $\mathfrak{A}_{\mathrm{L}}$ is its set of units, i.e. invertible elements. Let A_{L} be the group (or a subgroup) of units of $\mathfrak{A}_{\mathrm{L}}$ and let G_A denote a topological group isomorphic to A_{L}. By definition, G_A is a topological linear Lie group since A_{L} is endowed with a Lie bracket.([15] def. 5.32) Construct the complexified group $G_A^{\mathbb{C}}$. The plan is to model $\mathfrak{A}_{\mathrm{L}}$ by a certain C^*-algebra of functions $\mathrm{F} : G_A^{\mathbb{C}} \to L_B(\mathcal{H})$ (to be specified below). Of course this algebra is not likely to be equivalent to $\mathfrak{A}_{\mathrm{L}}$. But in practice one doesn't know $\mathfrak{A}_{\mathrm{L}}$ explicitly anyway: Invariably, one starts with some symmetries that characterize a system, identifies the associated group (which is typically a Lie group), and constructs $\mathfrak{A}_{\mathrm{L}}$ from there. So we might as well make this assumption:

Assumption 2.1 *The C^*-algebra that characterizes a quantum system can be modeled by a certain space of integrable functions $\mathrm{F} : G_A^{\mathbb{C}} \to L_B(\mathcal{H})$ where $G_A^{\mathbb{C}}$ is a topological linear Lie group whose Lie algebra possesses a triangular decomposition*[2] *and the Hilbert space $\mathcal{H}$ furnishes a suitable representation of $G_A^{\mathbb{C}}$.*

According to the assumption, quantization (partly) corresponds to identifying a topological linear Lie group, constructing its relevant representations, and building a suitable space of integrable functions. But generically, $G_A^{\mathbb{C}}$ is non-compact so it is not possible to directly extract measurable objects. So the assumption covers the first two points of kinematic input but does not address the Born rule. Fortunately,

[2]We add this qualifier as it will simplify the discussion of representations. But more importantly the decomposition characterizes physically relevant quantum numbers. It is not significantly restrictive from a physics perspective, because it includes all finite-dimensional and Kac-Moody Lie algebras.

measurement *is* possible with *locally compact* topological groups, and this underlies our entire construction.

We require, then, some rationale to obtain locally compact topological groups from $G^{\mathbb{C}}_A$. Let $G^{\mathbb{C}}_{A,\Lambda} := \{G^{\mathbb{C}}_{A,\lambda}, \lambda \in \Lambda\}$ represent a family of locally compact topological Lie groups $G^{\mathbb{C}}_{A,\lambda}$ indexed by continuous homomorphisms $\lambda : G^{\mathbb{C}}_A \to G^{\mathbb{C}}_{A,\lambda}$. We will be purposely nonspecific about the set Λ, because it depends on the particular quantum system under consideration. But in general it represents constraints, state preparation/observation, or any other system particulars that one must specify to implement the Born rule. The point is, $G^{\mathbb{C}}_A$ inherits a Lie bracket structure from the quantum C^*-algebra that can only be glimpsed as a member of $G^{\mathbb{C}}_{A,\Lambda}$ through observation/measurement.

This leads to the second assumption:

Assumption 2.2 *A query*[3] *of a quantum system corresponds to a homomorphism* $\lambda : G^{\mathbb{C}}_A \to G^{\mathbb{C}}_{A,\lambda}$ *where* $G^{\mathbb{C}}_{A,\lambda}$ *is a* <u>*locally compact*</u> *topological linear Lie group.*

Example 2.1 *A good example is the familiar Feynman path integral. Here* $G^{\mathbb{C}}_A$ *is the group (under point-wise addition) of Gaussian*[4] *pointed paths* $X_a \ni (x, t_a) \to (M, m_a)$ *where* $t_a \in \mathbb{R}$*,* $x(t_a) = m_a \in M$*, and* M *is some manifold.* X_a *is an infinite-dimensional abelian topological group when endowed with a suitable topology. The corresponding path integral over* X_a *is a formal object. But as soon as one imposes a constraint on the loose ends of the paths, for example* $x(t_b) = m_b$ *which 'pins' them to a single point, the group 'localizes' to a finite-dimensional group* $X_{a,b}$*.*[5] *Being a finite-dimensional topological vector space, it is automatically locally compact: The corresponding path integral can now be explicitly evaluated. There are of course many other 'constraints' that one can impose on a given system. These constitute the set* Λ*, and a particular choice of* $\lambda \in \Lambda$ *leads to a particular evaluation of the path integral over* X_a*.*

Finally, to be economical, we suppose dynamics are modeled by *inner* automorphisms of $\mathfrak{A}_{\mathrm{L}}$. It is natural to expect the algebra to include system dynamics since it contains a linear Lie group. After all, the system presumably evolves independent of any external input. This leads to our last assumption:

Assumption 2.3 *The dynamics of a closed quantum system are governed by continuous, time-dependent unitary* <u>*inner*</u> *automorphisms of the* C^**-algebra.*

[3]By 'query' we mean any observation one may perform that leads to a measurable quantity.

[4]By Gaussian paths we mean the pointed paths are characterized by a mean and covariance.

[5]To see this, parametrize the space of Gaussian pointed paths by mean and covariance. Fixing the loose end-point fixes the mean, and the covariance is then parametrized by points in M. Consequently, the moduli space of pointed paths with both end-points fixed is congruent to M.

2.2 Representations

Suppose a family $G^{\mathbb{C}}_{A,\Lambda}$ that governs some quantum system has been identified. The first order of business in the quantization program is to find all relevant representations (reps) $\rho : G^{\mathbb{C}}_{A,\lambda} \to L(\mathcal{H})$ for all $\lambda \in \Lambda$ where $L(\mathcal{H})$ denotes the set of linear operators on $\mathcal{H}$. Keeping in mind the important qualifiers indicated by sub/superscripts in $G^{\mathbb{C}}_{A,\lambda}$, we will often simply write G in this subsection to indicate a locally compact topological Lie group for notational clarity.

2.2.1 Induced reps

Our preliminary goal is to determine and interpret the reps $\varrho'^{(r)} : \mathfrak{G} \to L(\mathcal{V}^{(r)})$ furnished by $\mathfrak{G}$-modules $\mathcal{V}^{(r)}$ and labeled by r. Start with the triangular decomposition of the Lie algebra $\mathfrak{G}$;

$$\mathfrak{G} = \mathfrak{G}_+ \oplus \mathfrak{G}_0 \oplus \mathfrak{G}_+ \tag{2.1}$$

where

$$\begin{aligned}
&[\mathfrak{G}_0, \mathfrak{G}_0] = 0 \\
&[\mathfrak{G}_+, \mathfrak{G}_+] \subseteq \mathfrak{G}_0 \\
&[\mathfrak{G}_\pm, \mathfrak{G}_0 \oplus \mathfrak{G}_\pm] \subseteq \mathfrak{G}_\pm \ .
\end{aligned} \tag{2.2}$$

In a dominant-integral lowest/highest-weight representation, the Cartan subalgebra $\mathfrak{G}_0$ defines *potential* ‘charges’ and ‘ground states’ through a weight-space decomposition where ‘charge’ corresponds to a weight in the basis of fundamental weights and ‘ground state’ corresponds to the lowest/highest weight. These are *potential* identifications, because we have not yet determined that the quantum framework will respect the Lie algebra structure.

Let $\varrho' : \mathfrak{G} \to L(\mathcal{V})$ be a representation with $\mathcal{V}$ a $\mathfrak{G}$-module. The triangular decomposition of the algebra induces a decomposition of $\mathcal{V}$ by

$$\mathcal{V} = \bigoplus_w \mathcal{V}_{(w)} \,, \quad \mathcal{V}_{(w)} := \{\boldsymbol{v} \in \mathcal{V} : \varrho'(\mathfrak{h}_i)\boldsymbol{v} = w_i \boldsymbol{v}\} \,, \quad i \in \{1, \ldots, \mathrm{rank}(G)\} \tag{2.3}$$

where $\mathfrak{h}_i \in \mathfrak{G}_0$ and $w = \{w_1, \ldots, w_{\mathrm{rank}(G)}\}$ is a weight in the Dynkin basis composed of complex eigenvalues. Accordingly, representations are partially characterized by $\mathrm{rank}(G)$ symmetry charges.

In the weight decomposition of *finite-dimensional* $\mathcal{V}$, there is a distinguished subspace $\mathcal{V}_{(w_+)} \subset \mathcal{V}$ associated with a maximal weight w_+ such that

$$\varrho'(\mathfrak{g}_+)\boldsymbol{v} = 0, \quad \forall \mathfrak{g}_+ \in \mathfrak{G}_+ \tag{2.4}$$

for all $\boldsymbol{v} \in \mathcal{V}_{(w_+)}$. The same can be said for minimal weights w_-. For simplicity, we will assume the dynamical system is invariant under the inner automorphism $\mathfrak{G}_- \leftrightarrow \mathfrak{G}_+$, so there is no physical distinction between minimal and maximal weight. We might

as well follow mathematics convention and confine our attention to maximal weight modules.[6]

In particular, if ϱ' is *irreducible*, then there is only one maximal weight (now called the highest weight) and $\mathcal{V}_{(w_+)}$ is one-dimensional possessing a unique (up to scalar multiplication) highest-weight vector $\boldsymbol{v}_{w_+}$. In this case module $\mathcal{V}$, now denoted $\mathcal{V}_{w_+}$, is called a highest-weight module, and it is generated by acting on $\boldsymbol{v}_{w_+}$ with combinations of 'lowering operators' $\mathfrak{g}_- \in \mathfrak{G}_-$.

Finite-dimensional highest-weight irreducible representation (irrep) modules have three important properties: (i) if the highest weight is dominant-integral, then $\mathcal{V}_{w_+}$ possesses a positive-definite hermitian inner product, (ii) the module also furnishes an irrep for the connected component of G by exponentiation of ϱ', and (iii) its highest-weight vector $\boldsymbol{v}_{w_+}$ is a good candidate for a quantum ground state. Unfortunately, highest-weight $\mathcal{V}_{w_+}$ will not be finite-dimensional in general for all algebras possessing a triangular decomposition.

Infinite-dimensional highest-weight modules are generated by analogy: Choose a highest-weight eigenvector of $\mathfrak{G}_0$ and act on it by all combinations of lowering operators. The problem is the resulting representation is not irreducible in general. Consequently, its physical interpretation is problematic. Of course, one can construct an irreducible quotient Verma module, but being infinite-dimensional it doesn't necessarily furnish a representation of the group G which is amenable to physical interpretation. Fortunately there is a work-around — induced representations.

There is a distinguished subalgebra $\mathfrak{G}_c \subseteq \mathfrak{G}$; its maximal compact subalgebra. Let $\mathcal{V}_{(\mu)} \subset \mathcal{V}_{w_+}$ denote the submodule generated by $\mathfrak{G}_c$ acting on a dominant-integral highest-weight vector $\boldsymbol{v}_{w_+}$. The submodule $\mathcal{V}_{(\mu)}$ furnishes an irrep of $\mathfrak{G}_c$. And, since w_+ is a highest weight, $\mathcal{V}_{(\mu)}$ is an invariant subspace with respect to the parabolic subalgebra $\mathfrak{P} := \mathfrak{G}_+ \cup \mathfrak{G}_c$, i.e. $\bar{\varrho}'(\mathfrak{P})\mathcal{V}_{(\mu)} \subseteq \mathcal{V}_{(\mu)}$ with $\bar{\varrho}'$ a subrepresentation of ϱ'. This can be seen by using the triangular decomposition and the fact that $\mathfrak{G}_0 \subseteq \mathfrak{G}_c$. Moreover, $\bar{\varrho}'$ exponentiates to an irrep $\bar{\varrho}$ of a parabolic subgroup $P \subset G$ since $\mathcal{V}_{(\mu)}$ is finite-dimensional. Evidently, the span of $\mathcal{V}_{(\mu)}$ represents 'parabolic invariant states' cyclicly generated by $\boldsymbol{v}_{w_+}$.

With the pertinent parabolic subgroup identified, construct the principal coset bundle $(G, X, \breve{pr}, P)$ and its associated vector bundle $(\mathcal{V}, X, pr, \mathcal{V}_{(\mu)}, P)$ where the base space is a submanifold of the homogeneous coset space $X := G/P$ and $\breve{pr}$ (respectively pr) denotes the principal (respectively vector) bundle projection.

Following the standard method, an induced rep labeled by r is defined in terms of equivariant maps $\breve{\psi}^r \in L^2(G, \mathcal{V}^{(r)}_{(\mu)})$ by

$$\mathrm{Ind}_P^{G^{(r)}} = \{\breve{\psi}^r \in L^2(G, \mathcal{V}^{(r)}_{(\mu)}) \,|\, \breve{\psi}^r(g\,p) = \bar{\varrho}(p^{-1})\breve{\psi}^r(g)\} \tag{2.5}$$

where $p \in P$ and the continuous map $\bar{\varrho} : P \to L(\mathcal{V}^{(r)}_{(\mu)})$ is a dominant-integral highest-weight irrep. Similarly, an induced unitary rep (urep) is defined in terms of *normalized*

[6]However, there may be interesting physics associated with dynamical systems that are not invariant under $\mathfrak{G}_- \leftrightarrow \mathfrak{G}_+$, and this case deserves investigation.

equivariant maps $\breve{\psi}^r \in L^2(G, \mathcal{V}^{(r)}_{(\mu)})$ by

$$\mathrm{UInd}_P^{G^{(r)}} = \{\breve{\psi}^r \in L^2(G, \mathcal{V}^{(r)}_{(\mu)}) \,|\, \breve{\psi}^r(g\,p) = N(p)\bar{\varrho}(p^{-1})\breve{\psi}^r(g)\} \tag{2.6}$$

where $p \in P$, the normalization $N^2(p) := \triangle_P(p)/\triangle_G(p)$ with $\triangle_G(g) = |\det Ad_G(g)|$ the modular function of G, and now the continuous map $\bar{\varrho} : P \to L(\mathcal{V}^{(r)}_{(\mu)})$ is unitary.

Construct the Whitney sum bundle for all relevant reps (ureps) constructed from the basic representation modules $\mathcal{V}^{(r)}_{(\mu)}$ labeled by r

$$\begin{aligned} \mathcal{W}_{\mathcal{V}} &:= (\bigoplus_{r=r_1}^{r_{max}} \mathcal{V}^{(r)}, X, pr, \bigoplus_{r=r_1}^{r_{max}} \mathcal{V}^{(r)}_{(\mu)}, P) \\ &=: \big(\mathcal{W}, X, pr, \mathcal{W}_{(\boldsymbol{\mu})}, P\big) \end{aligned} \tag{2.7}$$

where $\boldsymbol{\mu} := (\mu^{(r_1)}, \ldots, \mu^{(r_{max})})$. Note that $\mathcal{W}_{(\boldsymbol{\mu})}$ *may be infinite dimensional but we will assume that it is Hilbert and separable.*

The induced rep $\rho : G \to L\left(L^2(G, \mathcal{W}_{(\boldsymbol{\mu})})\right)$ is furnished by

$$\mathrm{Ind}_P^G := \bigoplus_r \mathrm{Ind}_P^{G^{(r)}} \;. \tag{2.8}$$

The representation can be expressed as

$$(\rho(g)\breve{\psi})(g_o) = \breve{\psi}(g^{-1}g_o) =: \breve{\psi}_g(g_o) \tag{2.9}$$

where $g_o, g \in G$.

It is important to stress *these induced representations are not necessarily irreducible.* Fortunately, parabolically-induced irreps are much studied by mathematicians and we refer the reader to the literature.

2.2.2 Hilbert space of states

Elements of UInd_P^G are square-integrable maps characterized by the right action of P which induces a change of basis in $\mathcal{W}_{(\boldsymbol{\mu})}$. But the choice of basis in $\mathcal{W}_{(\boldsymbol{\mu})}$ is arbitrary to begin with. Therefore, physically relevant $\breve{\psi}$ should be invariant under the *right action of* P. This is just the statement of gauge invariance in the bundle framework.

Recall that $\breve{\psi} \in \mathrm{UInd}_P^G$ and $\psi \in L^2(X, \mathcal{W})$ can be identified by $\breve{\psi}(g) = g^{-1} \circ \psi(x)$ with the conditions $\breve{pr}(g) = x = gx_0$ where x_0 is a choice of origin in X.[7] If a canonical local section σ_i on the principal bundle is chosen relative to a local trivialization $\{U_i, \varphi_i\}$, then $\breve{\psi}$ and ψ are *canonically* related, and we can identify $\psi(\cdot) \equiv \breve{\psi}(\sigma_i(\cdot))$.[16] In other words, $\psi \equiv \sigma_i^*\breve{\psi}$. Explicitly, for $x \in U_i \subset X$, the representative of $\psi(x)$ relative to the local trivialization is $\psi(x) = (x, \boldsymbol{v}_{w_g})$ where $\boldsymbol{v}_{w_g} = \breve{\psi}(g)$. This canonical

[7] Because the principal and vector bundles are associated, g is both a point in $\breve{pr}^{-1}(x) \in G$ and an admissible map $g : \mathcal{W}_{(\boldsymbol{\mu})} \to pr^{-1}(x) \in \mathcal{W}$ ([16] pg. 367).

identification is tantamount to a choice of basis for each fiber $\mathcal{W}_x$ given by $\boldsymbol{e}_x = \sigma_i(x)\boldsymbol{e}_{(\mu)}$ where $\boldsymbol{e}_{(\mu)}$ is a basis of $\mathcal{W}_{(\boldsymbol{\mu})}$.

Note that $\breve{pr}(g\sigma_i(x)) = g\,\breve{pr}(\sigma_i(x)) = gx$ so $g\sigma_i(x)$ must be a point in the fiber over gx, i.e. $g\sigma_i(x) = \sigma_i(gx)p$ for some $p \in P$. Hence, using canonical local sections relative to a given local trivialization yields a canonical induced representation on $\mathcal{W}$;

$$\begin{aligned}
(\rho(g)\psi)(x) &= (\rho(g)\breve{\psi})(\sigma_i(x)) \\
&= \breve{\psi}(g^{-1}\sigma_i(x)) \\
&= \breve{\psi}(\sigma_i(g^{-1}x)p) \\
&= N(p)\bar{\varrho}(p^{-1})\sigma_i^*\breve{\psi}(g^{-1}x) = N(p)\bar{\varrho}(p^{-1})\psi(g^{-1}x) \\
&\qquad\qquad =: N(p)\bar{\varrho}(p^{-1})\psi_g(x) \,.
\end{aligned} \tag{2.10}$$

Remark that p depends on both x and g, and $\bar{\varrho}(p)$ can be interpreted as a gauge transformation about which much more can be said ([17] appx. A.1).

Use this urep to define an induced $*$-homomorphism $\pi_x : L_B(\mathcal{H}) \to L_B(\mathcal{W}_x)$ by

$$(\pi_x(\rho(g)))\,\boldsymbol{v}_{w_g} := (\rho(g)\psi)(x) \tag{2.11}$$

where $(x, \boldsymbol{v}_{w_g})$ is the representative of $\psi(x)$ in a local trivialization.

Evidently, the action of G on ψ is arbitrary up to some $p \in P$. This ambiguity is directly related to the choice of basis (relative to $\mathcal{W}_{(\boldsymbol{\mu})}$) for each fiber $\mathcal{W}_x$; and, like $\breve{\psi}$, physical ψ are therefore invariant under the right action of P. If $\breve{\psi}$ represents a state of the quantum system and ρ is a urep, then the canonical relation between $\breve{\psi}$ and ψ allows to postulate that a physical quantum state can be represented by an equivalence class $[\psi]$ where the equivalence relation is $\psi(x) \sim \psi(xp)$. In this sense, the module $\mathcal{H} = L^2(X, \mathcal{W}) \subset \Gamma(X, \mathcal{W})$, which furnishes the induced ureps of G, contains the physical Hilbert space comprised of 'gauge equivalent' ψ.

Use the hermitian inner product $(\cdot|\cdot)$ on $\mathcal{W}_{(\boldsymbol{\mu})}$ to construct a bundle metric on $\mathcal{W}$. Then equip $\mathcal{H}$ with the inner product induced from $\mathcal{W}$ and the quasi-invariant measure μ_P on X([10] pg. 138);

$$\langle\psi_1|\psi_2\rangle := \int_X (\breve{\psi}_1(g)|\breve{\psi}_2(g))_{\mathcal{W}_{(\boldsymbol{\mu})}}\, d\mu_P(gP) = \int_X (\psi_1(x)|\psi_2(x))_{\mathcal{W}_x}\, d\mu_P(x) \,. \tag{2.12}$$

Complete $\mathcal{H}$ with respect to the associated norm. Then $\mathcal{H}$ is Hilbert and it models the quantum Hilbert space. It must be emphasized, however, that this does not coincide with the *physical* Hilbert space which presumably furnishes the direct sum of all unitary irreps of gauge equivalent (with respect to P) maps $[\psi] \in L^2(X, \mathcal{W})$.

Definition 2.1 *The Hilbert space $\mathcal{H} = L^2(X, \mathcal{W})$ furnishes a direct sum of all ureps of G. The QM physical Hilbert space $\mathcal{H}_Q \subseteq \mathcal{H}$ is the subspace of gauge equivalent states $[\psi] \in L^2(X, \mathcal{W})$.*

It is important that the 'charges' and 'ground states' coming from the triangular decomposition of the Lie algebra carry over to the quantum Hilbert space. By construction, vectors in $\mathcal{W}_x \cong \mathcal{W}_{(\boldsymbol{\mu})}$ are labeled by 'charges' associated with G_c and the

quantum ground state ψ_0 can be defined by its representative $\psi_0(x) = (x, \boldsymbol{v}_{w_+})$ for all $x \in X$.

Now that we possess $G^{\mathbb{C}}_A$, its induced representations, and the furnishing Hilbert space $\mathcal{H} = L^2(X, \mathcal{W})$ properly normed and completed; it remains to construct a suitable C^*-algebra of functions to model $\mathfrak{A}_{\mathrm{L}}$.

2.3 Functional Mellin transform

The tool we use to construct the C^*-algebra is the functional Mellin transform. Functional Mellin transforms are a particular type of functional integral defined in [18]. Roughly, to define general functional integrals, we take the data $(G, \mathfrak{B}, G_\Lambda)$ with $\mathfrak{B}$ an associative Banach algebra and define a functional integral by a *family* of integral operators $\mathrm{int}_\Lambda : \mathbf{F}(G_\Lambda) \to \mathfrak{B}$ where $\mathbf{F}(G_\Lambda)$ is a family of spaces of integrable functions $f \in L^1(G_\lambda, \mathfrak{B})$ for all $\lambda \in \Lambda$. A brief introduction is given in appendix A.

In particular then, functional Mellin transforms are defined using the refined functional integral data $(G^{\mathbb{C}}_A, \mathfrak{C}^*, G^{\mathbb{C}}_{A,\Lambda})$ where $\mathfrak{C}^*$ is a *unital* C^*-algebra. For functional Mellin in the context of QM, we further stipulate $\mathfrak{C}^* \equiv L_B(\mathcal{H})$.

Definition 2.2 ([14]) *Let the map $\rho : G^{\mathbb{C}}_{A,\lambda} \to L_B(\mathcal{H})$ be a continuous, injective homomorphism, and $\pi_x : L_B(\mathcal{H}) \to L_B(\mathcal{W}_x)$ be the non-degenerate[8] $*$-homomorphism defined in (2.11). Define continuous functionals $\mathbf{F}(G^{\mathbb{C}}_A) \ni \mathrm{F} : G^{\mathbb{C}}_A \to L_B(\mathcal{H})$ equivariant under right-translations[9] according to $\mathrm{F}(gh) = \mathrm{F}(g)\rho(h)$. Then the functional Mellin transform $\mathcal{M}_\lambda : \mathbf{F}(G^{\mathbb{C}}_A) \to L_B(\mathcal{H})$ is defined by*

$$\mathcal{M}_\lambda [\mathrm{F}; \alpha] := \int_{G^{\mathbb{C}}_A} \mathrm{F}(gg^\alpha)\, \mathcal{D}_\lambda g = \int_{G^{\mathbb{C}}_A} \mathrm{F}(g)\rho(g^\alpha)\, \mathcal{D}_\lambda g \tag{2.13}$$

where $\alpha \in \mathbb{S} \subset \mathbb{C}$, $g^\alpha := \exp_G(\alpha \log_G g)$ and $\pi_x(\mathrm{F}(g)\rho(g^\alpha)) \in L_B(\mathcal{W}_x)$ and where the space of bounded linear operators $L_B(\mathcal{W}_x)$ is given the operator-norm topology. Denote the space of Mellin integrable functionals[10] by $\mathbf{F}_{\mathbb{S}}(G^{\mathbb{C}}_A)$.

Remark 2.1 The class of functional Mellin transforms defined here includes the crossed products of [10] as a special case. To relate the two, require ρ to be a strongly continuous unitary representation $U : G^{\mathbb{C}}_{A,\lambda} \to L_B(\mathcal{H})$. Then the integrated form of (π, U), called the crossed product and denoted $\pi \rtimes U(\mathrm{F} \circ \lambda)$, is equivalent to $\mathcal{M}_\lambda [\mathrm{F}; 1]$.

Now, define a norm on $\mathbf{F}_{\mathbb{S}}(G^{\mathbb{C}}_A)$ by $\|\mathrm{F}\| := \sup_\alpha \|\mathrm{F}\|_\alpha$ where

$$\|\mathrm{F}\|_\alpha := \sup_\lambda \|\mathcal{M}\,[f(g_\lambda); \alpha]\,\| < \infty\,, \qquad \alpha \in \mathbb{S}\,. \tag{2.14}$$

Assume that $\mathbf{F}_{\mathbb{S}}(G^{\mathbb{C}}_A)$ can be completed w.r.t. this (or some other suitably defined norm). Then

[8] Non-degenerate here means $\pi_x(L_B(\mathcal{H}))(\mathcal{W}_x)$ is dense in $\mathcal{W}_x$.

[9] This prescription is for left-invariant Haar measures. For right-invariant Haar measures impose equivariance under left-translations.

[10] Technically, the term ‘functional’ refers to a map from some vector space to its underlying scalar field. Since $G^{\mathbb{C}}_{A,\lambda}$ may be abelian and $L_B(\mathcal{H}) \cong \mathbb{C}$, use of the term in this case is strictly correct. It is appropriate then to use the term for the more general cases since it does not lead to confusion.

Proposition 2.1 ([14] prop. 4.2) $\mathbf{F}_{\mathbb{S}}(G^{\mathbb{C}}_A)$ *is a* C^**-algebra such that* $\|\mathrm{F}^*\|_\alpha = \|\mathrm{F}\|_\alpha$ *when endowed with an involution defined by* $\mathrm{F}^*(g^{1+\alpha}) := \mathrm{F}(g^{-1-\alpha})^*\Delta(g^{-1})$ *and suitable topology.*

So according to our first assumption, the space of Mellin integrable functionals $\mathbf{F}_{\mathbb{S}}(G^{\mathbb{C}}_A)$ models the C^*-algebra that characterizes the physical properties of a quantum system.

The utility of functional Mellin transforms is they realize $*$-representations of $\mathbf{F}_{\mathbb{S}}(G^{\mathbb{C}}_A)$ under suitable conditions.([14] corr. 4.2) For example, if $\mathfrak{C}^*$ is commutative then $\mathcal{M}_\lambda$ is a $*$-representation for all $\alpha \in \mathbb{S}$. On the other hand, if $\mathfrak{C}^*$ is non-commutative but $G^{\mathbb{C}}_A$ is abelian, then $\mathcal{M}_\lambda$ is a $*$-representation for $\alpha \in \mathbb{R}\cap\mathbb{S}$ if ρ is unitary or $\alpha \in i\mathbb{R}\cap\mathbb{S}$ if ρ is real. In the extreme case of non-commutative $\mathfrak{C}^*$ and non-abelian $G^{\mathbb{C}}_A$, then $\mathcal{M}_\lambda$ is a $*$-representation only for $\alpha = 0$ or $\alpha = 1$ and ρ unitary. We denote these separate cases by a single symbol $\mathcal{R}^{(\alpha)}_\lambda$ where α must be determined by context.

Using (2.9) and definition (2.11), the explicit realization of $\mathcal{R}^{(\alpha)}_\lambda(\mathrm{F}) \in L_B(\mathcal{H})$ in terms of $\pi_x(\mathrm{F}(g)) \in L_B(\mathcal{W}_x)$ is given by (for $\alpha \neq 0$)

$$\begin{aligned}
\left(\mathcal{R}^{(\alpha)}_\lambda(\mathrm{F})\,\psi\right)(x) &:= \pi_x\left(\mathcal{R}^{(\alpha)}_\lambda(\mathrm{F})\right)\,\psi(x)\\
&= \pi_x\left(\int_{G^{\mathbb{C}}_A} \mathrm{F}(g)\rho(g^\alpha)\,\mathcal{D}_\lambda g\right)\psi(x)\\
&= \int_{G^{\mathbb{C}}_A} \pi_x\left(\mathrm{F}(g)\right)\left((\pi_x\circ\rho)(g^\alpha)\right)\psi(x)\,\mathcal{D}_\lambda g\\
&= \int_{G^{\mathbb{C}}_A} \pi_x\left(\mathrm{F}(g)\right)\left(\rho(g^\alpha)\psi\right)(x)\,\mathcal{D}_\lambda g\\
&= \int_{G^{\mathbb{C}}_A} \pi_x\left(\mathrm{F}(g)\right)\left(\rho(g^\alpha)\breve{\psi}\right)(\sigma_i(x))\,\mathcal{D}_\lambda g\\
&= \int_{G^{\mathbb{C}}_A} \pi_x\left(\mathrm{F}(\sigma_i(x)^{1/\alpha}g)\right)\breve{\psi}(g^{-\alpha})\,\mathcal{D}_\lambda g
\end{aligned} \tag{2.15}$$

where left-invariance of the Haar measure was used in the last line. This yields explicit transition amplitudes

$$\langle\psi_1|\psi_2\rangle^{(\alpha)}_\lambda = \langle\psi_1|\mathcal{R}^{(\alpha)}_\lambda(\mathrm{F})\,\psi_1\rangle = \int_X (\psi_1(x)|\pi_x(\mathcal{R}^{(\alpha)}_\lambda(\mathrm{F}))\psi_1(x))_{\mathcal{W}_x}\,d\mu_P(x)\ .$$

It is important, of course, to construct $\mathcal{R}^{(\alpha)}_\lambda$ that are irreducible $*$-representations:

Proposition 2.2 *If* ρ *is an irrep, then* $\mathcal{R}^{(\alpha)}_\lambda$ *is an irrep for suitable* α.

Proof: First, by corollary 4.2 of [14], $\mathcal{R}^{(\alpha)}_\lambda$ is a $*$-representation. Next, let $\mathcal{S}$ be a

closed subspace of $\mathcal{H}$ invariant under ρ. Then for $\psi \in \mathcal{S}$ and $\varphi \in \mathcal{S}^{\perp}$, if $f(g_\lambda)\psi \in \mathcal{S}$,

$$\begin{aligned} \langle \mathcal{R}^{(\alpha)}_\lambda(\mathrm{F})\,\psi|\varphi\rangle &= \langle\left(\int_{G^{\mathbb{C}}_{A,\lambda}} f(g_\lambda)\rho(g^\alpha_\lambda)\,d\mu(g_\lambda)\right)\psi|\varphi\rangle \\ &= \int_{G^{\mathbb{C}}_{A,\lambda}} \langle (f(g_\lambda)\rho(g^\alpha_\lambda))\,\psi|\varphi\rangle\,d\mu(g_\lambda) \\ &= 0 \end{aligned} \tag{2.16}$$

where the last line follows because $f(g_\lambda)\rho\,(g^\alpha_\lambda))\,\psi \in \mathcal{S}$ for all $g_\lambda \in G^{\mathbb{C}}_{A,\lambda}$ by assumption. Hence, direct sums are preserved and therefore irreducible $\rho \Rightarrow$ irreducible $\mathcal{R}^{(\alpha)}_\lambda$. $\square$

The conclusion is functional Mellin offers a practical means to effect quantization given irreducible $\rho : G^{\mathbb{C}}_{A,\lambda} \to \mathfrak{C}^*$. In the context of QM, we will insist that ρ is induced from P, identify $\mathfrak{C}^* \equiv L_B(\mathcal{H})$, and equip $L_B(\mathcal{H})$ with the operator-norm topology. With this understood, the kinematical framework is nearly complete — it remains to interpret the set of continuous homomorphisms Λ.

2.4 Observation/Measurement

Although the act of observation/measurement is sometimes interpreted as non-unitary evolution — and hence dynamical in nature — in this scheme it is more naturally interpreted as kinematical.

We have seen that the functional $\mathrm{F} \in \mathbf{F}_{\mathbb{S}}(G^{\mathbb{C}}_A)$ corresponds to an entire family of functions $f \in L^1(G^{\mathbb{C}}_{A,\lambda}, L_B(\mathcal{H}))$ for each $\lambda \in \Lambda$ representing a set of homomorphisms onto measurable topological groups. It is easy to imagine that the physical quantum state of a macroscopic measuring device (which of course cannot be known exactly) that actualizes some 'observable'[11] is modeled by a suitable family of functions.

Furthermore, the convolution products in $\mathbf{F}_{\mathbb{S}}(G^{\mathbb{C}}_A)$ and $L^1(G^{\mathbb{C}}_{A,\lambda}, L_B(\mathcal{H}))$ are equivalent by definition, but their respective norms are not. Our choice of norm on $\mathbf{F}_{\mathbb{S}}(G^{\mathbb{C}}_A)$ — along with the fact that the convolution product and involution are only defined within each $L^1(G^{\mathbb{C}}_{A,\lambda}, L_B(\mathcal{H}))$ — renders it a direct sum $\mathbf{F}_{\mathbb{S}}(G^{\mathbb{C}}_A) = \bigoplus_{\lambda\in\Lambda} L^1(G^{\mathbb{C}}_{A,\lambda}, L_B(\mathcal{H}))$. Therefore, a 'query', which picks out a single λ, induces a projection.

So the measurement process gets a topological interpretation: Performing a measurement and thereby actualizing an observable corresponds to a particular projection of $\mathbf{F}_{\mathbb{S}}(G^{\mathbb{C}}_A)$ onto a locally compact copy $L^1(G^{\mathbb{C}}_{A,\lambda}, L_B(\mathcal{H}))$.[12] Precisely which projection is effected cannot be known. According to our last assumption, the system evolution

[11] As usual, an observable is a self-adjoint element $\mathrm{O} \in \mathbf{F}_{\mathbb{S}}(G^{\mathbb{C}}_A)$.

[12] We do not mean to imply that this projection has any causal effect on physical reality: We are in the Heisenberg picture so the system's initial wave function remains ontological/objective while the observable representing a measurement is epistemic/subjective. We do mean to imply that a *non-locally compact* C^*-algebra describes a pre-measured quantum system, and measurement (which corresponds to a particular λ) is only given meaning in the context of a *locally compact* C^*-algebra.

is generated by $G_A^{\mathbb{C}}$. So any subsequent measurement will of course be referred to $L^1(G_{A,\lambda}^{\mathbb{C}}, L_B(\mathcal{H})) \subset \mathbf{F}_{\mathbb{S}}(G_A^{\mathbb{C}})$ unless *external*[13] interaction dynamics takes the system out of this subspace.

Example 2.2 *Return to the Feynman path integral example. Recall that observing a point-to-point transition leads to a 'localization' $\lambda : X_a \to X_{a,b}$ to some finite-dimensional group $X_{a,b}$ — typically $X_{a,b} \cong \mathbb{R}^3$. But there is no guarantee the identity in X_a is mapped to the identity in $X_{a,b}$.*[14] *In other words, in the typical case, there is no preferred origin in $\mathbb{R}^3$ until an observation/measurement selects one. Once selected, the group $X_{a,b}$ with its preferred origin continues to govern the <u>closed</u> system evolution. However, external interaction necessarily implies a new $G_{A,\lambda}^{\mathbb{C}}$ along with the concomitant localization ambiguity. Alternatively, one can stipulate that identity elements are mapped to identity elements. Then the localization ambiguity should be interpreted as an ambiguity of the vacuum $\boldsymbol{v}_{w_+}$; in which case specifying λ includes a choice of vacuum.*

Essentially, the topological aspect of the model supplies a family of Hilbert spaces. The family represents a lack of knowledge; not of the system but of the measuring "ruler". Once a measurement has been made, it is given comparative meaning (that is, it can be compared to subsequent measurements) through a specific representation of the associated observable $\mathcal{M}_\lambda[\mathrm{O};\alpha]$ furnished by the Hilbert space based on the locally compact group $G_{A,\lambda}^{\mathbb{C}}$ and its associated vacuum that were selected by the measurement. This offers a topological replacement for wave-function collapse.

2.5 Quantum Hilbert module

The previous subsections can be efficiently organized and expressed under the rubric of Hilbert C^*-modules.[7]

Suppose we know the topological Lie group G and algebra $\mathfrak{A}_{\mathrm{L}}$ of some quantum system. Construct the vector bundle $(\mathcal{V}, X, pr, \mathcal{V}_{(\mu)}, P)$ and its associated principal bundle $(G, X, \breve{pr}, P)$ where $X = G/P$, the typical fiber $\mathcal{V}_{(\mu)}$ is Hilbert and possibly infinite-dimensional, and P is some chosen parabolic subgroup.

Call $\mathcal{E} \rightleftharpoons \mathfrak{Q}^*$ a quantum Hilbert C^*-module with linear[15] space $\mathcal{E} = \mathrm{Hom}_C(G, \mathcal{V}_{(\mu)})$ and algebra $\mathfrak{Q}^* = \mathbf{F}_{\mathbb{S}}(G)$. Here $\mathbf{F}_{\mathbb{S}}(G)$ is understood to be defined in terms of the functional Mellin transform. Identifying $\mathcal{E}$ with $L^2(X, \mathcal{V})$ in the same manner discussed in section 2.2.2, the Hilbert module $L^2(X, \mathcal{V}) \rightleftharpoons \mathbf{F}_{\mathbb{S}}(G)$ underlies the kinematic backdrop of the quantum system. But quantization requires one more step: Transition amplitudes between states in $\mathcal{E}$ only acquire meaning through a 'localization' $\lambda : G \to G_\lambda$ where G_λ is locally compact. Then $\mathcal{E}$ and $\mathfrak{Q}^*$ acquire explicit representations $\mathcal{E}_\lambda = L^2(X, \mathcal{V}_\lambda)$ and $\mathfrak{Q}^*_\lambda = \mathbf{F}_{\mathbb{S}}(G_\lambda)$ through *induced* representations of G_λ.

[13] Since a closed system is supposed to evolve according to a known $G_{A,\lambda}^{\mathbb{C}}$, it takes something outside the system to induce a new localization $\widetilde{\lambda} : G_A^{\mathbb{C}} \to G_{A,\widetilde{\lambda}}^{\mathbb{C}}$.

[14] This is reflected in the functional integral by the left-invariance of the Haar measure.

[15] The continuous homomorphisms are required to map the identity in G to the origin in $\mathcal{V}_{(\mu)}$.

So the *quantization* of a system characterized by $\mathcal{E} \rightleftharpoons \mathfrak{Q}^*$ is modeled by a *family* of Hilbert C^*-modules $\mathcal{E}_\Lambda \rightleftharpoons \mathbf{F}_{\mathbb{S}}(G_\Lambda)$. This framework allows one to work at the abstract level $L_B(\mathcal{H})$ as opposed to the concrete realization $L_B(\mathcal{V}_x) = \pi_x(L_B(\mathcal{H}))$. Of course, it is important to have both levels at one's disposal.

3 System Evolution

Recall that ρ is a representation of $G^{\mathbb{C}}_{A,\lambda}$, and it is not defined on $G^{\mathbb{C}}_A$. However, it is cumbersome and messy to keep indicating the λ dependence by always writing $\rho(g_\lambda)$. So for this entire section we will write simply $\rho(g)$.

3.1 Hamiltonians

We want to construct continuous, time-dependent unitary inner automorphisms on $\mathbf{F}_{\mathbb{S}}(G^{\mathbb{C}}_A)$ of the form $\mathrm{E}^{-i\mathrm{H}(t)}$ where E is defined in terms of the $*$-convolution that represents multiplication and $\mathrm{H}(t) := i\mathrm{Log}(\mathrm{E}^{-i\mathrm{H}(t)})$ is self-adjoint. The functional Log of $\mathrm{F} \in \mathbf{F}_{\mathbb{S}}(G^{\mathbb{C}}_A)$ is defined by[14]

$$(\mathrm{Log}\,\mathrm{F}^{-1})_\lambda := \frac{d}{d\alpha}\mathcal{M}_\lambda\left[\mathrm{E}^{-\mathrm{F}};\alpha\right]\Big|_{\alpha\to 0^+} \tag{3.1}$$

if the limit exists. Since $\mathrm{F} = \mathrm{E}^{-i\mathrm{H}(t)}$ is unitary,

$$i\mathrm{H}(t) = \frac{d}{d\alpha}\mathcal{M}_\lambda\left[\mathrm{E}^{-\mathrm{E}^{-i\mathrm{H}(t)}};\alpha\right]\Big|_{\alpha\to 0^+} . \tag{3.2}$$

Note that $\mathrm{H}(t) \in L(\mathcal{H})$ need not be bounded so it doesn't belong to $\mathbf{F}_{\mathbb{S}}(G^{\mathbb{C}}_A)$, and generically its Mellin transform will not be a $*$-representation. Nevertheless, for self-adjoint $\mathrm{H}(t)$, $\mathrm{E}^{-i\mathrm{H}(t)} \in \mathbf{F}_{\mathbb{S}}(G^{\mathbb{C}}_A)$ and $\mathcal{R}^{(\alpha)}_\lambda\left(\mathrm{E}^{-i\mathrm{H}(t)}\right) \in L_B(\mathcal{H})$.

These unitary inner automorphisms are supposed to be generated by a one-parameter subgroup $h(\mathbb{R}) \subset G^{\mathbb{C}}_A$. So representing $\mathrm{E}^{-i\mathrm{H}(t)}$ via Mellin transform requires finding a suitable Mellin integrand $\mathrm{F}(h(t))$. Off hand, one might think to use the exponential function $\exp_{G^{\mathbb{C}}_{A,\lambda}}$ on the group , but this would only give access to Hamiltonians generated by $\mathfrak{G}^{\mathbb{C}}_{A,\lambda}$.

Instead, take $\mathfrak{h}_{\mathrm{U}}(t) \in \mathrm{U}(\mathfrak{G}^{\mathbb{C}}_A)$ for each $t \in \mathbb{R}$ to be self-adjoint where $\mathrm{U}(\mathfrak{G}^{\mathbb{C}}_A)$ is the universal enveloping algebra of $\mathfrak{G}^{\mathbb{C}}_A$ and define

$$e^{-i\mathfrak{h}_{\mathrm{U}}(t)} := \sum_{n=0}^{\infty}\frac{1}{n!}(-i\mathfrak{h}_{\mathrm{U}}(t))^n . \tag{3.3}$$

Since ρ' extends to $\mathrm{U}(\mathfrak{G}^{\mathbb{C}}_{A,\lambda})$, the functional $e^{-i\rho'(\mathfrak{h}_{\mathrm{U}}(t))} =: e^{-iH(t)}$ is in $L_B(\mathcal{H})$, and it makes sense to take $\mathrm{F}(h(t)) = e^{f(dh(t))} = e^{-iH(t)}$ with $dh : \mathbb{R} \to \mathfrak{G}^{\mathbb{C}}_A$ and suitable $f : \mathfrak{G}^{\mathbb{C}}_A \to \mathrm{U}(\mathfrak{G}^{\mathbb{C}}_A)$. Remark that $\pi_x(e^{-iH(t)})$ is realized as exponentiated left-invariant differential operators on $\psi(x)$ as expected.

Evidently, the C^*-enveloping algebra of $\mathrm{U}(\mathfrak{G}^{\mathbb{C}}_A)$ is isomorphic to $\mathfrak{A}_{\mathrm{L}}$, and we see why $G^{\mathbb{C}}_{A,\Lambda}$ is so basic to the entire construction.

3.2 Dynamics

A first obvious remark is that nontrivial dynamics is only possible for non-commutative $L_B(\mathcal{H})$ and non-abelian $G^{\mathbb{C}}_A$.

Adopting our third assumption, quantum dynamics of a closed system is generated by a continuous, time-dependent inner automorphism $\mathrm{F} \mapsto h(t)\,\mathrm{F}\,h(t)^{-1} =: \mathrm{Ad}(h(t))\mathrm{F}$ with $\mathrm{F} \in \mathbf{F}_{\mathbb{S}}(G^{\mathbb{C}}_A)$ and unitary $h(t) \in G^{\mathbb{C}}_A$. Together with ρ, this induces an adjoint action $Ad : G^{\mathbb{C}}_{A,\lambda} \to \mathrm{Aut}\,(L_B(\mathcal{H}))$ by

$$\begin{aligned}
\mathcal{R}^{(\alpha)}_\lambda\left(\mathrm{Ad}(h(t))\mathrm{E}^{-\mathrm{F}}\right) &= \mathcal{R}^{(\alpha)}_\lambda\left(h(t)\mathrm{E}^{-\mathrm{F}}h^{-1}(t)\right) \\
&= \rho(h(t))\,\mathcal{R}^{(\alpha)}_\lambda\left(\mathrm{E}^{-\mathrm{F}}\right)\,\rho(h^{-1}(t)) \\
&= Ad(h(t))F^{-\alpha}_\lambda \\
&=: F^{-\alpha}_\lambda(t)
\end{aligned} \tag{3.4}$$

where the functional complex power[14] is defined by $F^{-\alpha}_\lambda := \mathcal{R}^{(\alpha)}_\lambda(\mathrm{E}^{-\mathrm{F}}) \in L_B(\mathcal{H})$, and $\mathrm{E}^{-\mathrm{F}} \in \mathbf{F}_{\mathbb{S}}(G^{\mathbb{C}}_A)$.

Equivalently, if $h(t)$ is differentiable,

$$\frac{d\mathrm{F}(t)}{dt} = \frac{d}{dt}\mathrm{Ad}(h(t))\,\mathrm{F} = -i\mathrm{Ad}'_e(\mathfrak{h}_{\mathrm{U}}(h(t)))\mathrm{F}(t) =: \mathrm{ad}(-i\mathfrak{h}_{\mathrm{U}}(h(t)))\mathrm{F}(t) \tag{3.5}$$

where $dh(t)/dt = -i\mathfrak{h}_{\mathrm{U}}(h(t)) = -iL'_{h(t)}\mathfrak{h}_{\mathrm{U}}(h(0))$ and $h(0) = e$ is the identity element. In other words, $\mathbf{F}_{\mathbb{S}}(G^{\mathbb{C}}_A)$ models the Lie bracket structure ostensibly possessed by $\mathfrak{A}_{\mathrm{L}}$ through the derivative of the induced adjoint action.

Using Magnus' expansion[19], $h(t)$ can be written (for suitable t)

$$h(t) = e^{-i\widetilde{\mathfrak{h}}_{\mathrm{U}}(t)} \tag{3.6}$$

such that

$$\frac{d\widetilde{\mathfrak{h}}_{\mathrm{U}}(t)}{dt} = \sum_{n=0}^{\infty}\frac{B_n}{n!}\,\mathrm{ad}^n\left(\mathfrak{h}_{\mathrm{U}}(t)\right)\widetilde{\mathfrak{h}}_{\mathrm{U}}(t) \tag{3.7}$$

where B_n are Bernoulli numbers and the map $\mathrm{ad}^n\left(\mathfrak{h}_{\mathrm{U}}(t)\right)$ is defined recursively by $\mathrm{ad}^0\left(\mathfrak{h}_{\mathrm{U}}(t)\right)\widetilde{\mathfrak{h}}_{\mathrm{U}}(t) := \widetilde{\mathfrak{h}}_{\mathrm{U}}(t)$ and $\mathrm{ad}^n\left(\mathfrak{h}_{\mathrm{U}}(t)\right)\widetilde{\mathfrak{h}}_{\mathrm{U}}(t) := \mathrm{ad}^1\left(\mathfrak{h}_{\mathrm{U}}(t)\right)ad^{n-1}\left(\mathfrak{h}_{\mathrm{U}}(t)\right)\widetilde{\mathfrak{h}}_{\mathrm{U}}(t)$. This leads to the Heisenberg equation

$$\frac{dF^{-\alpha}_\lambda(t)}{dt} = ad(-i\widetilde{H}(t))F^{-\alpha}_\lambda(t) = -i\left[\widetilde{H}(t), F^{-\alpha}_\lambda(t)\right]\ . \tag{3.8}$$

Meanwhile, $\mathrm{Ad}(h(t))$ also induces an automorphism $Ad : G^{\mathbb{C}}_{A,\lambda} \to \mathrm{Aut}(\rho'(\mathfrak{G}^{\mathbb{C}}_A))$ such that $\rho'(\mathfrak{g}) \mapsto Ad(h(t))\rho'(\mathfrak{g}) =: \rho'(\mathfrak{g}(t))$. Again, if $h(t)$ is differentiable,

$$\frac{d\rho'(\mathfrak{g}(t))}{dt} = -i\left[\widetilde{H}(t), \rho'(\mathfrak{g}(t))\right]\ . \tag{3.9}$$

Note that $\rho'(\mathfrak{g}(t)) \in L(\mathcal{H})$, not being bounded, is not the image of a time-dependent observable in $\mathbf{F}_{\mathbb{S}}(G_A^{\mathbb{C}})$. Nevertheless, if $\mathfrak{g}(t)$ is (anti)self-adjoint, it possesses a time-dependent spectrum that represents evolution of the kinematical description in the sense that the parabolic decomposition of $\mathfrak{G}_A^{\mathbb{C}}$ and the consequent induced representation ρ are time-dependent in general.

Put $U(t) := \rho(h(t))$ and $\psi(t) := U(t)\psi$.[16] Note that $U(t)$ depends implicitly on λ. As usual, unitarity supplies the connection between the Heisenberg and Schrödinger pictures

$$\langle\phi|\mathcal{R}_\lambda^{(\alpha)}(\mathrm{F}(t))\psi\rangle = \langle\phi|U(t)^{-1}\,\mathcal{R}_\lambda^{(\alpha)}(\mathrm{F})\,U(t)|\psi\rangle = \langle\phi(t)|\mathcal{R}_\lambda^{(\alpha)}(\mathrm{F})|\psi(t)\rangle\ , \tag{3.10}$$

and the dynamics give rise to time-dependent transition amplitudes

$$\begin{aligned}\langle\psi_1|\mathcal{R}_\lambda^{(\alpha)}(\mathrm{F}(t))\psi_1\rangle &= \int_X (\psi_1(x)|\pi_x(\mathcal{R}_\lambda^{(\alpha)}\mathrm{F}(t))\psi_1(x))_{\mathcal{W}_x}\, d\mu_P(x)\\ &= \int_X (\psi(x;t)|\pi_x(\mathcal{R}_\lambda^{(\alpha)}\mathrm{F})\psi(x;t))_{\mathcal{W}_x}\, d\mu_P(x)\ .\end{aligned} \tag{3.11}$$

By construction, these transition amplitudes are gauge covariant. That is, they are invariant under $\psi(x) \to \psi(xp)$ for all $p \in P$.

3.3 Resolvent of a conservative Hamiltonian

Let $\phi_{\mathfrak{h}_\mathrm{U}} \in \mathrm{Hom}_C(\mathbb{C}, G_A^{\mathbb{C}})$ denote a continuous one-parameter subgroup of $G_A^{\mathbb{C}}$ generated by a *time-independent* self-adjoint Hamiltonian $\mathfrak{h}_\mathrm{U} \in \mathrm{U}(\mathfrak{G}_A^{\mathbb{C}})$. In particular, for some initial time t_0, $\rho(\phi_{\mathfrak{h}_\mathrm{U}}(i\mathbb{R}))$ represents the evolution operator for $t \geq t_0$ *and* $t \leq t_0$. That is, we interpret $\rho(\phi_{\mathfrak{h}_\mathrm{U}}(t)) =: e^{-H\,t} \in L_B(\mathcal{H})$, for all $t \in i\mathbb{R}$, as the union of time-forward and time-reversed strongly continuous semi-groups that yield system evolution from the starting time t_0.

The functional Mellin transform can be used to associate various observables in $\mathbf{F}_{\mathbb{S}}(G_A^{\mathbb{C}})$ with the evolution operator $U(t) := e^{-H\,t}$ under the conditions that render it a $*$-representation. For example, $\mathcal{R}_\lambda^{(\alpha)}(\mathrm{E}^{-(\mathrm{H}-z\mathrm{Id})})$ gives the resolvent of H for $\alpha = 1$, a suitable choice of λ, and $z \notin \sigma(H)$. This follows readily from functional Mellin[14]

$$\begin{aligned}\mathcal{R}_\Gamma^{(\alpha)}(\mathrm{E}^{-(\mathrm{H}-z\mathrm{Id})}) &= \int_{\phi_{\mathfrak{h}_\mathrm{U}}(i\mathbb{R})} e^{-(H-zId)g}\,\rho(g^\alpha)\,d\nu(g_\Gamma)\\ &= \int_0^{i\infty} e^{-(H-zId)g}\,g^\alpha\,d\nu(g_\Gamma) + \int_0^{-i\infty} e^{-(H-zId)g}\,g^\alpha\,d\nu(g_\Gamma)\\ &= \left\{\begin{array}{l}(zId-H)_\Gamma^{-\alpha}\ ,\ \ \Im(z)<0\\ (zId-H)_\Gamma^{-\alpha}\ ,\ \ \Im(z)>0\end{array}\right\}\ ,\qquad \alpha\in(0,\infty)\ .\end{aligned} \tag{3.12}$$

[16] One should be careful not to misinterpret $\psi(t)$. It represents a time-dependent element of $\mathcal{H}$, but it is *not* a function of t unless the base space of the associated fiber bundle is augmented to $\mathbb{R}_+ \times X$.

The measure on the first line is chosen to be $\nu(g_\Gamma) = \log(g)/\Gamma(\alpha)$. The second line uses $\phi_{\mathfrak{h}_\mathrm{U}}(i\mathbb{R}) \cong i\mathbb{R}_+ \cup -i\mathbb{R}_+$. And the third line uses left-invariance of the Haar measure and the fact that $(H - zId)$ is invertible for $z \notin \sigma(H)$.

If H is positive-definite, one can go further and define the functional power[14] of a time-independent Hamiltonian observable

$$(\mathrm{H}^{-\alpha})_\lambda := H_\lambda^{-\alpha} = \int_{\phi_{\mathfrak{h}_\mathrm{U}}(i\mathbb{R})} e^{-Hg}\,\rho(g^\alpha)\,d\nu(g_\lambda)\,, \quad \alpha \in \mathbb{S}\,. \tag{3.13}$$

Recall that α is restricted according to the nature of the representation ρ, and it is not usually allowed to take values $\Re(\alpha) \le 0$.

Example 3.1 *A familiar but instructive example is the elementary kernel Δ^{-1} of the Laplacian on $\mathbb{R}^n$, explicitly realized on $\mathcal{W}_x \cong \mathbb{R}^n$.*

The degrees of freedom associated with a free particle on $\mathbb{R}^n$ are encoded by a continuous map $x : \mathbb{R}_+ \to \mathbb{R}^n$ which dictates $\lambda : G_A^\mathbb{C} \to \mathbb{R}_+ \times \mathbb{R}^n$ with $P = \mathbb{R}^n$ and $X = \mathbb{R}_+$.[17] The right action of the generators of $\mathbb{R}^n$ on x is by multiplication so the unitary right action of P amounts to multiplication by a phase. Consequently, physical states are rays in $\mathcal{H}$.

The elementary kernel for point-to-point boundary conditions is given by

$$\begin{aligned} \boldsymbol{K}(\mathrm{x}_a, \mathrm{x}_{a'}) &:= (\mathrm{x}_{a'}|\Delta_H^{-1}|\mathrm{x}_a) \\ &= (\mathrm{x}_{a'}|\mathcal{M}_H\left[\mathrm{E}^{-\Delta};1\right]|\mathrm{x}_a) \\ &= \int_{\phi_\Delta(i\mathbb{R})} (\mathrm{x}_{a'}|e^{-g\Delta}\,g^\alpha|\mathrm{x}_a)\,\mathcal{D}_H g\bigg|_{\alpha=1} \end{aligned} \tag{3.14}$$

where $\mathrm{x}_a := x(t_a) \equiv \sigma_i^* \breve{x}(t_a)$ and the subscript H (which has noting to do with the operator H above) denotes the choice of normalized Haar measure.

The expectation $(\mathrm{x}_{a'}|e^{-g\Delta}\,g^\alpha|\mathrm{x}_a)$ can be interpreted as a (time-forward or time-reversed) transition element on the principal bundle $(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+, \breve{pr}, \mathbb{R}^n)$ — otherwise known as an equivariant group propagator. Alternatively, as discussed previously, it can be formulated on the associated vector bundle and represented (up to gauge equivalence) as a functional integral over the abelian topological group X_a of $L^{2,1}(\mathbb{R}_+, \mathbb{R}_+ \times \mathbb{R}^n)$ pointed paths $(x, \mathrm{x}_a) : \mathbb{R}_+ \to \mathbb{R}_+ \times \mathbb{R}^n$ with $x(0) = \mathrm{x}_a$ compactly supported on $[t_a, t_b] \subset \mathbb{R}_+$. The pointed paths are characterized by mean paths satisfying $D\bar{x} = 0$ and covariance (symmetric quadratic form)

$$-\,Q(x_1, x_2) = \frac{1}{2}\{\langle x_1, D\,x_2\rangle + \langle x_2, D\,x_1\rangle\} - B(\bar{x}_1, \bar{x}_2) \tag{3.15}$$

where

$$\langle x_1, D\,x_2\rangle = \int_{t_a}^{t_b} (x_1(t)|D\,x_2(t))_t\,dt\,, \tag{3.16}$$

[17] We have augmented the base space $\{e\}$ to $\mathbb{R}_+ \times \{e\}$ in order to interpret x as a function of t as discussed in the previous footnote.

with $D = d^2/dt^2$ and B is a symmetric boundary term associated with the constraint $x(t_b) = \mathrm{x}_b$.[20],[21]

The calculation of the functional integral is standard and yields the equivariant propagator (for the time-forward case)

$$(\mathrm{x}_{a'}|e^{-g\Delta}\,g^{\alpha}|\mathrm{x}_a) = \theta(g)\,g^{\alpha-n/2}e^{-\pi|\mathrm{x}_{a'}-\mathrm{x}_a|^2/g}\;. \tag{3.17}$$

Consequently, the elementary kernel is given by (with Haar-normalization)

$$\begin{aligned}\boldsymbol{K}(\mathrm{x}_a,\mathrm{x}_{a'}) &= \int_{\phi_\Delta(i\mathbb{R})}(\mathrm{x}_{a'}|e^{-g\Delta}\,g^{\alpha}|\mathrm{x}_a)\;\mathcal{D}_H g\Big|_{\alpha=1}\\ &= \int_{\mathbb{R}_+}e^{-\pi|\mathrm{x}_{a'}-\mathrm{x}_a|^2/t}\,t^{\alpha-n/2}\;dt\Big|_{\alpha=1}\;,\qquad n/2-1\in\langle 0,\infty\rangle\\ &= \pi^{1-n/2}\,\Gamma(n/2-1)|\mathrm{x}_{a'}-\mathrm{x}_a|^{2-n}\;,\;\;n>2\;.\end{aligned} \tag{3.18}$$

The integral is not defined for $n \leq 2$. But the fundamental strip can be extended to the left of the origin by regularizing; $\not{\mathrm{E}}^{-\Delta} := (\mathrm{E}^{-\Delta} - \mathrm{E}^{-\mathrm{Id}})$.[14] *For $\mathbb{R}^2$ this gives*

$$\boldsymbol{K}(\mathrm{x}_a,\mathrm{x}_{a'}) = (\mathrm{x}_{a'}|\pi(\mathcal{M}_H\left[\not{\mathrm{E}}^{-\Delta};1\right])|\mathrm{x}_a) = -2\log|\mathrm{x}_{a'}-\mathrm{x}_a|\;, \tag{3.19}$$

while for $\mathbb{R}^1$, $\boldsymbol{K}(\mathrm{x}_a,\mathrm{x}_{a'}) = -2\pi|\mathrm{x}_{a'}-\mathrm{x}_a| + 2\pi$.

Given the existence of a functional power of some operator $H \in L(\mathcal{H})$ defined by (3.13), for suitable conditions on the spectrum of H one can define the functional trace[14]

$$(\mathrm{Tr}\,\mathrm{H}^{-\alpha})_\lambda := \int_{\phi_{\,\mathfrak{h}_{\mathrm{U}}}(i\mathbb{R})}\mathrm{tr}\left(e^{-Hg}\rho(g^{\alpha})\right)\;d\nu(g_\lambda)\,,\qquad \alpha\in\mathbb{S} \tag{3.20}$$

where tr is the trace in $\mathcal{H}$. In generic cases, this integral requires regularization to be well-defined. Closely related to the functional trace is the functional determinant[14]

$$\begin{aligned}(\mathrm{Det}\,\mathrm{H}^{-\alpha})_\lambda &:= \int_{\phi_{\,\mathfrak{h}_{\mathrm{U}}}(i\mathbb{R})}\det\left(e^{-Hg}\,\rho(g^{\alpha})\right)\;d\nu(g_\lambda)\\ &= \int_{\phi_{\,\mathfrak{h}_{\mathrm{U}}}(i\mathbb{R})}e^{-\mathrm{tr}(Hg)}\det\left(\rho(g^{\alpha})\right)\;d\nu(g_\lambda)\\ &=: \mathcal{R}^{(\alpha)}_\lambda(\mathrm{E}^{-\mathrm{Tr(H)}})\,,\qquad \alpha\in\mathbb{S}\;.\end{aligned} \tag{3.21}$$

This too typically requires regularization.

Unfortunately, analogous *simple* objects are not available for time-dependent operators generated by time-dependent $\mathfrak{h}_{\mathrm{U}}(t)$. The reason is $G^{\mathbb{C}}_A$ can no longer be reduced to a simple one-parameter subgroup, so the time-dependent $H(t)$ can't be diagonalized by a time-independent basis in $\mathcal{H}$. Loosely speaking, one needs an infinite set of eigenfunctions; one for each $t \in i\mathbb{R}$. Consequently, the corresponding operators representing observables are full-blown functional integrals over (generally) non-abelian groups that are not easy to evaluate. Nonetheless, the full arsenal of functional integral techniques and methods still apply. Better yet, they apply within the rigorous C^*-algebraic setting.

4 Summary

The assumptions enumerated in section 2.1, along with the functional integration framework summarized in appendix A, provide a hybrid realization of the axioms of quantum mechanics incorporating both functional and algebraic constructs. The center-piece is a topological group: (i) its induced representations directly determine the Hilbert space of states, (ii) it indirectly determines the quantum C^*-algebra through the functional Mellin transform, (iii) it suggests a topological interpretation of the measurement process, and (iv) it generates the dynamics through the adjoint action on the C^*-algebra.

The topological group is the star, but the functional Mellin transform is the work-horse. Once the underlying topological group has been specified, functional Mellin simultaneously defines the quantum C^*-algebra and provides representations of its observables, transition amplitudes, traces, and determinants. There is sophisticated mathematical machinery surrounding non-commutative function spaces, and the expectation is that functional Mellin will benefit from this and perhaps lead to useful computational techniques and methods in quantum mechanics.

There is a particularly notable aspect of the construction; the economy of assumptions that lead to quantization. This is due to the leading role played by the topological group: it underlies both the kinematics and dynamics. It means that one can replace the notion of 'quantizing a classical system' with 'specifying a topological group'. There is no ambiguity associated with the latter, and the role of the correspondence principle is reversed — it now defines the 'classical system'. But, of course, there is no free lunch because one must still somehow determine the evolution observables and physical Hilbert space that yield the correct dynamics.

A Functional integration

This appendix is a very brief summary of [18].

Consider the data $(G, \mathfrak{B}, G_\Lambda)$ where G is a Hausdorff topological group, $\mathfrak{B}$ is a Banach space that may have additional algebraic structure, and $G_\Lambda := \{G_\lambda, \lambda \in \Lambda\}$ is a family of locally compact topological groups indexed by continuous homomorphisms $\lambda : G \to G_\lambda$. The rigorous $\mathfrak{B}$-valued integration theory associated with $\{G_\lambda, \lambda \in \Lambda\}$ is used to define and characterize functional integration on G.

Definition A.1 *Let ν be a left Haar measure on G_λ, and $L^1(G_\lambda, \mathfrak{B})$ be the Banach space of $\mathfrak{B}$-valued functions $f : G_\lambda \to \mathfrak{B}$ integrable with respect to ν. Let $\mathbf{F}(G)$ denote the space of integrable functionals $\mathrm{F} : G \to \mathfrak{B}$.*

A family (indexed by Λ) of integral operators $\mathrm{int}_\Lambda : \mathbf{F}(G) \to \mathfrak{B}$ is defined by

$$\mathrm{int}_\lambda(\mathrm{F}) = \int_G \mathrm{F}(g)\mathcal{D}_\lambda g := \int_{G_\lambda} f(g_\lambda)\, d\nu(g_\lambda) \tag{A.1}$$

where $\mathrm{F} = f \circ \lambda$ with $f \in L^1(G_\lambda, \mathfrak{B})$ for all $\lambda \in \Lambda$. We say that F is integrable with respect to the integrator family $\mathcal{D}_\Lambda g$.

Further, if $\mathfrak{B}$ is an algebra, define the functional $*$-convolution by

$$(\mathrm{F}_1 * \mathrm{F}_2)_\lambda (g) := \int_G \mathrm{F}_1(\tilde{g})\mathrm{F}_2(\tilde{g}^{-1}g)\mathcal{D}_\lambda \tilde{g} \tag{A.2}$$

for each $\lambda \in \Lambda$.

For any given λ, the integral operator is linear and bounded according to

$$\|\mathrm{int}_\lambda(\mathrm{F})\| \leq \int_{G_\lambda} \|f(g_\lambda)\| \, d\nu(g_\lambda) = \|f\|_1 < \infty \, . \tag{A.3}$$

This suggests to define the norm $\|\mathrm{F}\| := \sup_\lambda \|\mathrm{F}\|_\lambda$ where

$$\|\mathrm{F}\|_\lambda := \int_G \|\mathrm{F}(g)\| \, \mathcal{D}_\lambda g \; := \int_{G_\lambda} \|f(g_\lambda)\| \, d\nu(g_\lambda) = \|f\|_1 < \infty \, . \tag{A.4}$$

The definition of $*$-convolution then implies

$$\begin{aligned} \|\mathrm{F}_1 * \mathrm{F}_2\|_\lambda &= \int_{G_\lambda}\int_{G_\lambda} \|f_1(\tilde{g}_\lambda) f_2(\tilde{g}_\lambda^{-1} g_\lambda)\| \, d\nu(\tilde{g}_\lambda, g_\lambda) \\ &= \int_{G_\lambda}\int_{G_\lambda} \|f_1(\tilde{g}_\lambda) f_2(g_\lambda)\| \, d\nu(\tilde{g}_\lambda) d\nu(g_\lambda) \\ &\leq \int_{G_\lambda}\int_{G_\lambda} \|f_1(\tilde{g}_\lambda)\|\|f_2(g_\lambda)\| \, d\nu(\tilde{g}_\lambda) d\nu(g_\lambda) \\ &= \|\mathrm{F}_1\|_\lambda \|\mathrm{F}_2\|_\lambda \end{aligned} \tag{A.5}$$

where the second line follows from left-invariance of the Haar measure and the last line follows from Fubini. Moreover, a similar computation (using left-invariance and Fubini) establishes $(\mathrm{F}_1 * \mathrm{F}_2) * \mathrm{F}_3 = \mathrm{F}_1 * (\mathrm{F}_2 * \mathrm{F}_3)$. Consequently, $\mathbf{F}(G)$ inherits the algebraic structure of $\mathfrak{B}$:

Proposition A.1 ([18], Prop. 2.2) *If $\mathfrak{B} \equiv \mathfrak{B}^*$ is a Banach $*$-algebra, then $\mathbf{F}(G)$ — endowed with a suitable topology and involution $\mathrm{F}^*(g) := \mathrm{F}(g^{-1})^* \Delta(g^{-1})$ and completed with respect to the norm $\|\mathrm{F}\| = \sup_\lambda \|\mathrm{F}\|_\lambda$ — is a Banach $*$-algebra, and int_λ is a $*$-homomorphism.*

Corollary A.1 *If $\mathfrak{B}$ is a C^*-algebra, then $\mathbf{F}(G)$ is C^*-algebra when completed w.r.t. the norm $\|\mathrm{F}\| = \sup_\lambda \|\mathrm{F}\|_\lambda$.*

Note that the products in $\mathbf{F}(G)$ and $L^1(G_\lambda, \mathfrak{B})$ are trivially equivalent by definition, but their respective norms are not. Our choice of norm on $\mathbf{F}(G)$ (along with the fact that the product and involution are only defined within each $L^1(G_\lambda, \mathfrak{B})$) renders it a direct sum $\mathbf{F}(G) = \bigoplus_{\lambda \in \Lambda} L^1(G_\lambda, \mathfrak{B})$. In this regard, a 'query' — which corresponds to a particular λ — induces a projection.

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